VIBRATIONAL INSTABILITY OF STATIONARY CONVECTION OF A NON-NEWTONIAN FLUID

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UDC 536.25:532.135

The stability of stationary convective flow of a non-Newtonian fluid to small perturbations, such as traveling thermal waves, is investigated.

Convective flow of a non-Newtonian fluid is investigated for an infinite vertical planar channel with isothermal walls having a constant temperature difference. The distance between the walls is 2h and the temperature difference is $2\Theta_0$. The rheological equation is

$$\tau_{ij} = -\delta_{ij}p + \eta(1 + aI)^{n-1} e_{ij}. \quad (1)$$

For n = 1 or $a = 0$ the rheological equation of an ordinary Newtonian fluid is obtained, while for $a \to \infty$ a transition occurs to the Ostwald power-law model.

At a constant temperature difference between the walls a stationary plane-parallel motion is generated in the channel with odd velocity profiles. Stationary flow was considered in detail in [1]. In the present paper we investigate the stability of such flow to monotonic hydrodynamic disturbances.

As is well known, a vibrational instability to perturbations such as traveling thermal waves is generated in a flow of a non-Newtonian fluid, starting with a Prandtl number equal to 11.4 (see [2]). With increasing Pr this instability becomes more dangerous. The same result was qualitatively obtained in [3] for a power-law fluid, where the flow stability was investigated in the effective viscosity approximation by the Galerkin method.

In the present paper we consider the stability of stationary convective flow with a large Prandtl number to planar "normal" velocity and temperature disturbances of the form $\exp(-\lambda t + ikz)$, where k is the wave number in the direction of the vertical z axis and λ is the decrement. An equation was derived in [1] for the amplitudes of normal disturbances. Denoting the amplitude of the perturbing stream function by $\Phi(x)$ and the amplitude of the temperature perturbation by $\vartheta(x)$, we obtain:

$$\lambda\Delta\Phi - ik\,\mathrm{Gr}\,H\Phi + M(\Phi^{\mathrm{IV}} + k^4\Phi) - 2Nk^2\Phi'' + 2Rk^2\Phi' + 2(L+R)\Phi''' + (L+R)'(\Phi'' + k^2\Phi) + \vartheta'' = 0, \quad (2)$$

$$\mathrm{Pr}^{-1}\Delta\vartheta + ik\,\mathrm{Gr}(T_0'\Phi - v_0\vartheta) = -\lambda\vartheta, \quad (3)$$

where $H \equiv v_0\Delta - v_0''$; $\Delta\Phi = \Phi'' - k^2\Phi$; $\Delta^2\Phi = \Phi^{\mathrm{IV}} - 2k^2\Phi'' + k^4\Phi$.

The following notation was used in these equations:

$$A \equiv 1 + \tilde{a}|v_0'|, \quad R \equiv \tilde{a}(n-1)(A^{n-2})'|v_0'|,$$
$$L \equiv 2(A^{n-1})', \quad M \equiv A^{n-1} + \tilde{a}(n-1)A^{n-2}|v_0'|,$$
$$N \equiv A^{n-1} - \tilde{a}(n-1)A^{n-2}|v_0'|.$$

The coordinate axes were chosen in such a manner that the x axis is perpendicular to the channel walls and the z axis is parallel to them and passes through the middle of the layer. The prime denotes differentiation with respect to x.

The following boundary conditions are satisfied at the solid isothermal walls of the channel:

$$\Phi = \Phi' = \vartheta = 0 \quad \text{for} \quad x = \pm 1. \quad (4)$$

All quantities in the equations derived are dimensionless. The units of distance, time, stream, and temperature functions are, respectively, h, $h^2\rho/\eta$, $\rho g\beta\Theta_0 h^3/\eta$, Θ_0. The dimensionless parameters of the problem are the Grashof and Prandtl numbers:

$$\mathrm{Gr} = \rho^2 g\beta\Theta_0 h^3/\eta^2, \quad \mathrm{Pr} = \eta/\chi\rho,$$

A. M. Gorki Perm State University. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 35, No. 2, pp. 320-325, August, 1978. Original article submitted June 20, 1977.

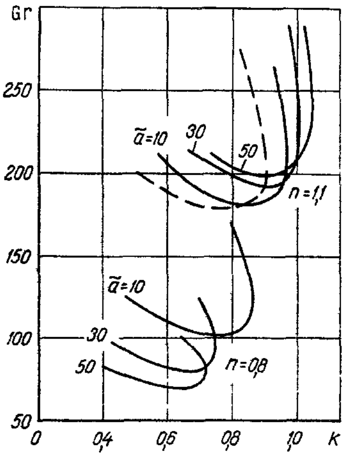

Fig. 1

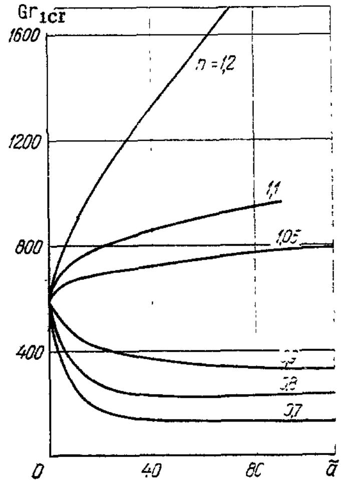

Fig. 2

Fig. 1. Lower sections of the neutral curves of vibrational instability of a stationary convective flow of a non-Newtonian fluid at Pr = 20. The dashed line is the neutral curve for a Newtonian fluid.

Fig. 2. The critical Grashof number Gr_{1cr} as a function of the dimensionless parameter $\tilde{a}$ for various values of n.

where v_0 and T_0 are the stationary velocity and temperature profiles.

The boundary-value problem (2)-(4) determines a spectrum of decrement eigenvalues λ and eigenfunctions Φ(x) and ϑ(x).

For certain values of the parameters n, $\tilde{a}$, Gr, Pr, and k the problem (2)-(4) was solved by the Runge–Kutta method with stepwise orthogonalization (the method is presented in detail in [4]). The lowest branch of the decrement spectrum was searched. The critical Grashof number as well as the phase velocity corresponding to a neutral disturbance were determined from the vanishing condition of the real part of the decrement Re(λ) = 0. We thus succeeded in constructing the neutral curve Gr(k) for fixed values of the remaining parameters.

The calculations were performed for Pr = 20. Figure 1 shows the neutral curves of vibrational instability for several values of the rheological parameters n and $\tilde{a}$. For comparison, the dashed line shows the neutral curve of a Newtonian fluid with the corresponding Prandtl number. The three lowest curves of the family correspond to a medium with pseudoplastic behavior (n = 0.8). As seen from Fig. 1, an increase in the value of $\tilde{a}$ leads to a decrease in the critical Grashof number, i.e., to flow destabilization. The same result is qualitatively obtained for monotonic hydrodynamic instability [1]. Besides, an increase in the $\tilde{a}$ value leads to a marked decrease in the critical wave number. The upper part of Fig. 1 shows neutral curves of a dilatational fluid (n = 1.1). Here, on the other hand, with increasing $\tilde{a}$ both the critical Grashof number and the critical wave number increase.

To determine the limits of stability in the preceding case of large Prandtl numbers we apply the method of asymptotic expansions in powers of the small parameter $Pr^{-1/2}$. Using this method, the stability of several convective flows of a Newtonian fluid was investigated [5].

We introduce the small parameter $\varepsilon = Pr^{-1/2}$ and represent Φ, ϑ, Gr, and λ in the form of the expansions

$$\Phi = \Phi_0 + \varepsilon\Phi_1 + \dots, \quad \lambda = \varepsilon\lambda_0 - \varepsilon^3\lambda_2 + \dots, \quad \vartheta = \vartheta_0 - \varepsilon\vartheta_1 + \dots, \quad Gr = \varepsilon\, Gr_1 + \dots \tag{5}$$

Substituting (5) in system (2), (3) and retaining only zeroth-order terms in ε, we obtain

$$M(\Phi_0^{IV} + k^4\Phi_0) - 2Nk^2\Phi_0'' + 2Rk^2\Phi_0' + 2(L+R)\Phi_0''' + (L+R)'(\Phi_0'' + k^2\Phi) + \vartheta_0' = 0, \tag{6}$$

$$ik\, Gr_1 (T_0' \Phi_0 - v_0\vartheta_0) = -\lambda_0\vartheta_0. \tag{7}$$

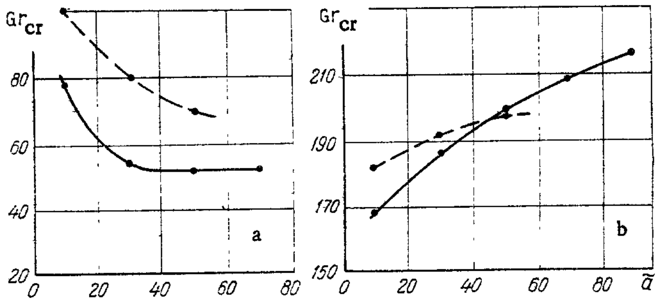

Fig. 3. The critical Grashof number as a function of the parameter $\tilde{a}$ for Pr = 20 and n = 0.8 (a); n = 1.1 (b). The solid line results from the asymptotic method and the dashed line from the method of stepwise integration of the amplitude equations.

It follows from Eq. (7) with account of the linearity of the basic temperature profile that $\vartheta_0 = \Phi_0/(u - v_0)$, where $u = \lambda_0/ik\,Gr_1$. Following substitution in the first equation, we obtain a zeroth-order boundary-value problem for Φ_0:

$$M(\Phi_0^{IV} + k^4\Phi) - 2Nk^2\Phi_0'' + 2Rk^2\Phi_0' + 2(L+R)\Phi_0''' + (L+R)'(\Phi_0'' + k^2\Phi) + \left(\frac{\Phi_0}{u - v_0}\right)'' = 0, \quad (8)$$

$$\Phi_0 = \Phi_0' = 0 \quad \text{for} \quad x = \pm 1. \quad (9)$$

The conjugate problem is hence constructed. This is done by standard procedures [2]. As a result, we obtain

$$(M\psi)^{IV} + Mk^4\psi - 2k^2(N\psi)'' - 2k^2(R\psi)' - 2(S\psi)''' + (S'\psi)'' + k^2S'\psi - \frac{\psi'}{u - v_0} = 0, \quad (10)$$

$$\psi = \psi' = 0 \quad \text{for} \quad x = \pm 1. \quad (11)$$

Here S = L + R.

The first-order equation in ε is the following

$$M(\Phi_1^{IV} + k^4\Phi_1) - 2Nk^2\Phi_1'' + 2Rk^2\Phi_1' + 2S\Phi_1''' + S'(\Phi_1'' + k^2\Phi) + \vartheta_1' = -\lambda\Delta\Phi_0 + ik\,Gr_1 H\Phi_0, \quad (12)$$

$$ik\,Gr_1(T_0'\Phi_1 - v_0\vartheta_1) + \lambda_0\vartheta_1 = -\Delta\vartheta_0. \quad (13)$$

Eliminating ϑ_1, we obtain an equation for the stream amplitude to first order:

$$M(\Phi_1^{IV} + k^4\Phi_1) - 2Nk^2\Phi_1'' + 2Rk^2\Phi_1' + 2(L+R)\Phi_1''' + (L+R)'(\Phi_1'' + k^2\Phi_1) + \left(\frac{\Phi_1}{u - v_0}\right)' = Q(\Phi_0), \quad (14)$$

where $Q(\Phi_0) = -ik\,Gr_1(u\Delta\Phi_0 - H\Phi_0) + \frac{1}{ik\,Gr_1}\left[\frac{\Delta\Phi_0}{(u-v_0)^2} + \frac{v_0''\Phi_0 + 2v_0'\Phi_0'}{(u-v_0)^3} + \frac{2v_0'^2\Phi_0}{(u-v_0)^4}\right]'$. The solvability condition of inhomogeneous equation (14) is the following:

$$\int_{-1}^{1} Q(\Phi_0)\psi dx = 0, \quad (15)$$

where ψ is the solution of conjugate problem (10), (11). An equation for Gr_1 is obtained from this condition:

$$Gr_1 = \left(\frac{j_1}{k^2 j_2}\right)^{\frac{1}{2}}, \quad (16)$$

where

$$j_1 = \int_{-1}^{1}\left[\frac{\Delta\Phi_0}{(u-v_0)^2} + \frac{v_0''\Phi_0 + 2v_0'\Phi'}{(u-v_0)^3} + \frac{2v_0'^2\Phi_0}{(u-v_0)^4}\right]\psi' dx;$$

$$j_2 = -\int_{-1}^{1}(u\Delta\Phi_0 - H\Phi_0)\psi dx.$$

Boundary-value problems (8), (9) and (10), (11) were integrated by the Runge – Kutta method. The eigenvalues were found by the chord method from the solutions of the conjugate problems. Since the phase velocity of the disturbance for large Pr is always larger than the maximum flow velocity, Eqs. (8), (10) contain no singularities.

Neutral curves $Gr_1(k)$ were calculated for n values equal to 0.7, 0.8, 0.9, 1.05, 1.1, 1.2, and for various values of $\tilde{a}$. Minimization with respect to the wave number k gives the critical value Gr_{1cr}. Figure 2 shows the dependence of the critical value Gr_{1cr} on the parameter $\tilde{a}$ for various n. The maximum value of $\tilde{a}$ up to which calculations were performed is 110; for n = 1.05 this value is 150. For n < 1 there occurs in these limits of $\tilde{a}$ a noticeable constancy of the value Gr_{1cr}. For a dilatational fluid (n > 1) this does not occur. For all n the value of Gr_{1cr} tends to 590 for $\tilde{a} \to 0$, corresponding to the well-known result for a Newtonian fluid.

The critical value of the Grashof number in the preceding case of large Pr is related to Gr_{1cr} by the equation

$$Gr_{cr} = [Gr_{1cr}(\tilde{a}, n)]/\sqrt{Pr}.$$

Thus, for large Pr, as well as in the case of a Newtonian fluid, the square root law $Gr_{cr} = C/Pr^{1/2}$ is satisfied, where the coefficient C depends on the rheological parameters. As seen from Fig. 2, for a pseudoplastic fluid in the range of large $\tilde{a}$ the coefficient C stops being dependent on $\tilde{a}$ and is determined by the index n only. While for n = 1 (a Newtonian fluid) C = 590, for n = 0.9, 0.8, 0.7 we have C = 325, 230, 130, respectively, indicating a significant reduction in stability.

It is interesting to compare the stability limit obtained by the asymptotic method with the results of numerical integration of the total amplitude equations of normal disturbances. Figure 3 shows this comparison for Pr = 20. The solid line shows the dependence $Gr_{cr} = Gr_{1cr}/\sqrt{20}$ and the dashed one is obtained from the critical points of the neutral curves shown in Fig. 1.

The results seem to be in qualitative agreement. The quantitative differences are related to the fact that the asymptotic results are used in the range of quite high Pr values [5]; in this sense Pr = 20 cannot be considered to be sufficiently large.

It follows from the results of this work and of [1] that for a non-Newtonian fluid with a rheological equation of type (1) the same instability mechanisms of a stationary plane-parallel convective flow occur as for a Newtonian fluid. The quantitative instability characteristics depend strongly on the rheological parameters.

NOTATION

h, half-width of the layer; Θ_0, half of the wall temperature difference; τ_{ij}, internal stress tensor; e_{ij}, deformation rate tensor; $I = \sqrt{(1/2)\dot{e}_{ij}\dot{e}_{ji}}$, second invariant of the deformation rate tensor; n, rheological index (positive and integer); a, rheological parameter; $\tilde{a}$, dimensionless rheological parameter; η, consistency coefficient; p, pressure; t, time; λ, time decrement; k, wave number; Φ, amplitude of normal disturbances of the stream function; ϑ, amplitude of normal temperature disturbances; v_0, stationary velocity distribution; T_0, stationary temperature distribution; ρ, fluid density; g, acceleration due to gravity; β, temperature coefficient of bulk expansion; χ, thermal diffusivity; and ψ, eigenfunction of the conjugate boundary-value problem.

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